SUBJECT: MATHEMATICS

PAPER-I, UNIT – IV: DIFFERENTIAL EQUATIONS, SEMESTER-II

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LINEAR DIFFERENTIAL EQUATION WITH CONSTANT COEFFICIENTS:

A differential equation of the form

$$a_n \frac{d^n y}{dx^n} + a_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_2 \frac{d^2 y}{dx^2} + a_1 \frac{dy}{dx} + a_0 y = Q(x) \ldots (1)$$

where $a_n , a_{n-1} , \ldots a_2 , a_1 , a_0$ are constants and Q(x) is either function of x or constant, is called a linear differential equation with constant coefficients of order n.

Using operator $D \; for \; \frac{d}{dx}$, equation (1) can be written as

$$(a_n D^n + a_{n-1} D^{n-1} + \cdots + a_2 D^2 + a_1 D + a_0)y = Q(x)$$

Or f(D)y = Q(x) ……(2)

If Q(x) = 0 then f(D)y =0 is called Homogeneous, otherwise Non-homogeneous.

Solution of homogeneous equation is called Complementary function of Non-homogeneous.

Also from equation (2), $y = \frac{1}{f(D)} Q(x)$ is Particular integral of Non-homogeneous.

So, complete solution of equation (2) is written as

COMPLETE SOLUTION = COMPLEMENTARY FUNCTION

+ PARTICULAR INTEGRAL

METHOD FOR FINDING THE COMPLEMENTARY FUNCTION

(1) In finding the complementary function, R.H.S. of the given equation is replaced by zero i.e. f(D)y =0
(2) Let $y = e^{mx}$ be the C.F. of f(D)y =0 i.e. replace D by m in f(D) and equate to zero.
(3) Then$(a_n m^n + a_{n-1} m^{n-1} + \cdots + a_2 m^2 + a_1 m + a_0) = 0$.

It is called **Auxiliary equation**. Let its roots are $m_1, m_2, \ldots m_n$.

Now three cases arise:

Case I : Roots, Real and Different. If the roots are distinct, then the C.F. is

$$y = C_1 e^{m_1 x} + C_2 e^{m_2 x} + ... + C_1 e^{m_n x}$$

Case II : Roots, Real and Equal. If all the roots are equal i.e. m then the C.F. is

$$y = (C_1 + C_2 x + C_3 x^2 + ... + C_n x^{n-1}) e^{mx}$$

Case III : Roots, Imaginary. If the roots are $\alpha \pm i\beta,$ then the solution will be

$$y = C_1 e^{(\alpha + i\beta)x} + C_2 e^{(\alpha - i\beta)x} = e^{\alpha x}.[C_1 e^{i\beta x} + C_2 e^{-i\beta x}]$$

$$= e^{\alpha x}[C_1(\cos\beta x + i\sin\beta x) + C_2(\cos\beta x - i\sin\beta x)]$$

$$= e^{\alpha x}[(C_1 + C_2)\cos\beta x + i(C_1 - C_2)\sin\beta x]$$

$$= e^{\alpha x}[A\cos\beta x + B\sin\beta x]$$

Ques. Solve: $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = 0$

Sol. Here, we have

$$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = 0 \quad \Rightarrow \quad (D^2 - 6D + 9)\ y = 0$$

A.E. is $m^2 - 6m + 9 = 0 \quad \Rightarrow \quad (m-3)^2 = 0 \quad \Rightarrow \quad m = 3, 3.$

$$C.F. = (C_1 + C_2 x)e^{3x}$$ Ans.

Ques. Find the general solution of the differential equation

$$\frac{d^5y}{dx^5} - \frac{d^3y}{dx^3} = 0$$

Sol. Here, we have

$$\frac{d^5y}{dx^5} - \frac{d^3y}{dx^3} = 0$$

$$\Rightarrow \quad D^5y - D^3y = 0 \quad \Rightarrow \quad (D^5 - D^3)y = 0 \quad \Rightarrow \quad D^3(D^2 - 1)y = 0$$

A.E. is $m^3(m^2 - 1) = 0 \quad \Rightarrow \quad m = 0,\ 0,\ 0,\ 1, -\ 1$

Hence, is solution is

$$y = (C_1 + C_2 x + C_3 x^2) + C_4 e^x + C_5 e^{-x}$$ Ans.

Ques. Solve: $\frac{d^2y}{dx^2}+4\frac{dy}{dx}+5y=0,$

Sol. Here the auxiliary equation is

$$m^2+4m+5=0 \qquad \Rightarrow \qquad m=\frac{-4\pm\sqrt{16-20}}{2}=-2\pm i$$

Its roots are $-2\pm i$

The complementary function is

$$y=e^{-2x}(A\cos x+B\sin x)$$

Rules to find particular integral:

(i) $\frac{1}{f(D)}e^{ax}=\frac{1}{f(a)}e^{ax}$

If $f(a)=0$ then $\frac{1}{f(D)}\cdot e^{ax}=x\cdot\frac{1}{f(a)}\cdot e^{ax}$

If $f'(a)=0$ then $\frac{1}{f(D)}\cdot e^{ax}=x^2\frac{1}{f''(a)}\cdot e^{ax}$

(ii) $\frac{1}{f(D)}x^n=[f(D)]^{-1}x^n$ Expand $[f(D)]^{-1}$ and then operate.

(iii) $\frac{1}{f(D^2)}\sin ax=\frac{1}{f(-a^2)}\sin ax$ and $\frac{1}{f(D^2)}\cos ax=\frac{1}{f(-a^2)}\cos ax$

If $f(-a^2)=0$, then $\frac{1}{f(D^2+a^2)}\sin ax=-\frac{x}{2a}\cdot\cos ax$

and $\frac{1}{f(D^2+a^2)}\cos ax=\frac{x}{2a}\sin ax$

(iv) $\frac{1}{f(D)}e^{ax}\cdot\phi(x)=e^{ax}\cdot\frac{1}{f(D+a)}\phi(x)$ where $\phi(x)$ is some functions of x.

(v) $\frac{1}{D+a}\phi(x)=e^{-ax}\int e^{ax}\cdot\phi(x)dx$

(vi) Symbol D stands for the operation of differential *i.e.*,

$$Dy=\frac{dy}{dx},\quad D^2y=\frac{d^2y}{dx^2}\ \ldots\ldots$$

$\frac{1}{D}$ stands for the operation of integration.

$\frac{1}{D^2}$ stands for the operation of integration twice.

Ques. Solve the following differential equation:

$$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} - 6y = \sin 3x + \cos 2x.$$

Sol. Here, we have

$$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} - 6y = \sin 3x + \cos 2x$$

$\Rightarrow$ $(D^2 + 5D - 6)y = \sin 3x + \cos 2x$

A.E. is $m^2 + 5m - 6 = 0$

$\Rightarrow$ $m^2 + 6m - m - 6 = 0$

$\Rightarrow$ $m(m+6) - 1(m+6) = 0$

$\Rightarrow$ $(m-1)\ (m+6) = 0$

$\Rightarrow$ $m = 1, -6$

$$C.F. = C_1 e^{x} + C_2 e^{-6x}$$

$$P.I. = \frac{1}{D^2 + 5D - 6}(\sin 3x + \cos 2x)$$

$$= \frac{1}{D^2 + 5D - 6}\sin 3x + \frac{1}{D^2 + 5D - 6}\cos 2x$$

$$= \frac{1}{-3^2 + 5D - 6}\sin 3x + \frac{1}{-2^2 + 5D - 6}\cos 2x$$

$$= \frac{1}{5D - 15}\sin 3x + \frac{1}{5D - 10}\cos 2x$$

$$= \frac{(D+3)}{5(D-3)(D+3)}\sin 3x + \frac{1(D+2)}{5(D-2)(D+2)}\cos 2x$$

$$= \frac{D+3}{5(D^2 - 9)}\sin 3x + \frac{D+2}{5(D^2 - 4)}\cos 2x$$

$$= \frac{D+3}{5(-3^2 - 9)}\sin 3x + \frac{D+2}{5(-2^2 - 4)}\cos 2x$$

$$= \frac{D+3}{5(-18)}\sin 3x + \frac{D+2}{5(-8)}\cos 2x$$

$$= \frac{(D+3)}{-90}(\sin 3x) - \frac{1}{40}(D+2)(\cos 2x)$$

$$= -\frac{1}{90}[3\cos 3x + 3\sin 3x] - \frac{1}{40}[-2\sin 2x + 2\cos 2x]$$

$$= -\frac{1}{30}(\cos 3x + \sin 3x) - \frac{1}{20}(\cos 2x - \sin 2x)$$

Complete solution is $y = C.F. + P.I.$

$\Rightarrow$ $$y = C_1 e^{x} + C_2 e^{-6x} - \frac{1}{30}(\cos 3x + \sin 3x) - \frac{1}{20}(\cos 2x - \sin 2x)$$

Ques. Solve: $\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} + 4\frac{dy}{dx} - 2y = e^{x} + \cos x$

Sol. We have, $(D^3 - 3D^2 + 4D - 2)y = e^{x} + \cos x$

A.E. is $m^3 - 3m^2 + 4m - 2 = 0$

$\Rightarrow (m-1)(m^2 - 2m + 2) = 0, i.e., m = 1,\ 1 \pm i$

$\therefore$ $$C.F. = C_1 e^{x} + (C_2 \cos x + C_3 \sin x)e^{x}$$

$$P.I. = \frac{1}{(D-1)(D^2 - 2D + 2)}e^{x} + \frac{1}{D^3 - 3D^2 + 4D - 2}\cos x$$

$$= \frac{1}{(D-1)(1-2+2)}e^{x} + \frac{1}{(-1)D - 3(-1) + 4D - 2}\cos x$$

$$= \frac{1}{(D-1)}e^{x} + \frac{1}{3D+1}\cos x$$

$$= e^{x}\frac{1}{D+1-1}.1 + \frac{3D-1}{9D^2 - 1}\cos x$$

$$= e^{x}\frac{1}{D}.1 + \frac{(-3\sin x - \cos x)}{-9-1}$$

$$= e^{x}.x + \frac{1}{10}(3\sin x + \cos x)$$

Hence, complete solution is

$$y = C_1 e^x + (C_2 \cos x + C_3 \sin x)e^x + x e^x + \frac{1}{10}(3\sin x + \cos x)$$

Ques. Solve: $\dfrac{d^2y}{dx^2} + y = \sin x \sin 2x$

Sol. We have,

$$\frac{d^2y}{dx^2} + y = \sin x \sin 2x$$

$$(D^2+1)y = \sin x \sin 2x$$

A.E. is $\quad m^2 + 1 = 0 \quad \Rightarrow \quad m = \pm i$

C.F. $= C_1 \cos x + C_2 \sin x$

P.I. $= \dfrac{1}{D^2+1}\sin x \sin 2x = \dfrac{1}{D^2+1}\dfrac{[\cos x - \cos 3x]}{2}$

$$= \frac{1}{2}\left[\frac{1}{D^2+1}\cos x - \frac{1}{D^2+1}\cos 3x\right]$$

$$= \frac{1}{2}\left[\frac{x}{2}\sin x - \frac{1}{-9+1}\cos 3x\right] = \frac{1}{2}\left[\frac{x}{2}\sin x + \frac{1}{8}\cos 3x\right]$$

$$= \frac{1}{16}[4x\sin x + \cos 3x]$$

Complete Solution is $y =$ C.F.+ P.I.

$$\Rightarrow \quad y = C_1 \cos x + C_2 \sin x + \frac{1}{16}(4x\sin x + \cos 3x)$$

Ques. Obtain the general solution of the differential equation

$$y'' - 2y' + 2y = x + e^x \cos x$$

Sol. We have, $y'' - 2y' + 2y = x + e^x \cos x$

A.E. is $\quad m^2 - 2m + 2 = 0 \quad \Rightarrow m = 1 \pm i$

$$C.F. = e^x (A\cos x + B\sin x)$$

$$P.I. = \frac{1}{D^2 - 2D + 2}x + \frac{1}{D^2 - 2D + 2}e^x \cos x$$

Where $I_1 = \frac{1}{D^2-2D+2}x = \frac{1}{2\left[1-D+\frac{D^2}{2}\right]}x = \frac{1}{2\left[1-\left(D-\frac{D^2}{2}\right)\right]}x$

$$= \frac{1}{2}\left[1-\left(D-\frac{D^2}{2}\right)\right]^{-1} x = \frac{1}{2}\left[1+\left(D-\frac{D^2}{2}\right)+...\right]x$$

$$= \frac{1}{2}\left[x+Dx-\frac{D^2}{2}x+...\right] = \frac{1}{2}[x+1]$$

and $I_2 = \frac{1}{D^2-2D+2}e^x \cos x$

$$= e^x \frac{1}{(D+1)^2-2(D+1)+2}\cos x = e^x \frac{1}{D^2+1}\cos x$$

$$= \frac{1}{2}xe^x \sin x \left[\text{If } f(-a^2)=0, \text{then } \frac{1}{f(D^2+a^2)}\cos ax = \frac{x}{2a}\sin ax\right]$$

$$y = C.F. + P.I.$$

$$= e^x(A\cos x + B\sin x) + \frac{1}{2}(x+1) + \frac{1}{2}xe^x \sin x.$$ Ans.

Ques. Solve: $(D^2-4D+4)y = x^3e^{2x}$

Sol. We have, $(D^2-4D+4)y = x^3e^{2x}$

A.E. is $m^2-4m+4=0 \quad \Rightarrow (m-2)^2=0 \quad \Rightarrow m=2,\ 2$

C.F. $=(C_1+C_2x)e^{2x}$

P.I. $= \frac{1}{D^2-4D+4}x^3.e^{2x} = e^{2x}\frac{1}{(D+2)^2-4(D+2)+4}x^3$

$$= e^{2x}\frac{1}{D^2}x^3 = e^{2x}.\frac{1}{D}\left(\frac{x^4}{4}\right) = e^{2x}.\frac{x^5}{20}$$

The complete solution is $y=(C_1+C_2x)e^{2x}+e^{2x}.\frac{x^5}{20}$ Ans

Ques. Solve $(D^4-1)y = e^x \cos x$

Sol. Here, we have

$$(D^4 - 1)y = e^x \cos x$$

A.E. is $m^4 - 1 = 0 \Rightarrow (m+1)(m-1)(m^2+1) = 0$

$\Rightarrow \quad m = -1,\ 1, +i, -i$

$$\text{C.F.} = C_1 e^x + C_2 e^x + (C_3 \cos x + C_4 \sin x)$$

$$\text{P.I.} = \frac{1}{D^4 - 1} e^x \cos x$$

$$= e^x \frac{1}{(D+1)^4 - 1} \cos x = e^x \frac{1}{D^4 + 6D^3 + 4D^2 + 6D} \cos x$$

$$= e^x \frac{1}{(-1)^2 + 6(-1)D + 4(-1) + 6D} \cos x$$

$$= e^x \frac{1}{1 - 6D - 4 + 6D} \cos x = -\frac{e^x \cos x}{3}$$

Complete solution is $y = C.F. + P.I.$

$$\Rightarrow y = C_1 e^{-x} + C_2 e^x + (C_3 \cos x + C_4 \sin x) - \frac{e^x \cos x}{3} \qquad \text{Ans.}$$

TO FIND THE VALUE OF $\frac{1}{f(D)} x^n \sin ax.$

Now $\quad \frac{1}{f(D)} x^n (\cos ax + i \sin ax) = \frac{1}{f(D)} x^n e^{iax} = e^{iax} \frac{1}{f(D+ia)} x^n$

$$\boxed{\frac{1}{f(D)} \cdot x^n \sin ax = \text{ Imaginary part of } e^{iax} \frac{1}{f(D+ia)} \cdot x^n}$$

$$\boxed{\frac{1}{f(D)} \cdot x^n \cos ax = \text{ Real part of } e^{iax} \frac{1}{f(D+ia)} \cdot x^n}$$

Ques. Solve the differential equation:

$$(D^2 + 2D + 1)y = x \cos x$$

Sol. $\quad (D^2 + 2D + 1)y = x \cos x$

Auxiliary equation is

$$m^2 + 2m + 1 = 0 \quad \Rightarrow \quad (m+1)^2 = 0 \quad \Rightarrow \quad m = -1, -1$$

$$\text{C.F. } = (C_1 + C_2 x)\, e^{-x}$$

$$\text{P.I. } = \frac{1}{(D+1)^2} x\cos x = \text{ Real part of } \frac{1}{(D+1)^2} x[\cos x + i\sin x]$$

$$= \text{Real part of } \frac{1}{(D+1)^2} x e^{ix} = \text{Real part of } e^{ix} \frac{1}{(D+i+1)^2} x$$

$$= \text{Real part of } e^{ix} \frac{1}{D^2 + 2(i+1)D + (i+1)^2} x = \text{Real part of } e^{ix} \frac{1}{D^2 + 2(1+i)D + 2i} x$$

$$= \text{Real part of } \frac{e^{ix}}{2i} \frac{1}{1 + \frac{1+i}{i} D + \frac{D^2}{2i}} x = \text{Real part of } \frac{e^{ix}}{2i}\left[1 + \frac{1+i}{i} D + \frac{D^2}{2i}\right]^{-1} x$$

$$= \text{Real part of } \frac{e^{ix}}{2i}\left[1 - \left(\frac{1+i}{i}\right) D + \ldots\right] x = \text{Real part of } \frac{1}{2i}(\cos x + i\sin x)\left[x - \frac{1+i}{i}\right]$$

$$= \text{Real part of } -\frac{1}{2}(i\cos x - \sin x)(x + i - 1)$$

$$= \text{Real part of } -\frac{1}{2}(i\cos x - \sin x)(x + i - 1) = \frac{1}{2}\sin x(x-1) + \frac{1}{2}\cos x$$

Complete solution is $y = C.F. + P.I.$

$$\Rightarrow \qquad y = (C_1 + C_2 x)e^{-x} + \frac{1}{2}(x-1)\sin x + \frac{1}{2}\cos x$$

Ques. Solve $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = x\sin x$

Sol. Auxiliary equation is $m^2 - 2m + 1 = 0 \Rightarrow m = 1,1$

$$C.F. = (C_1 + C_2 x)e^x$$

$$\text{P.I. } = \frac{1}{D^2 - 2D + 1} x \cdot \sin x \qquad (as\, e^{ix} = \cos x + i\sin x)$$

$$= \text{Im. part of } \frac{1}{D^2 - 2D + 1} x(\cos x + i\sin x) = \text{Im. part of } \frac{1}{D^2 - 2D + 1} x \cdot e^{ix}$$

$$= \text{ Imaginary part of } e^{ix} \frac{1}{(D+i)^2 - 2(D+i) + 1} \cdot x$$

$= \text{Imaginary part of } e^{ix} \dfrac{1}{D^2 - 2(1-i)D - 2i} \cdot x$

$= \text{Imaginary part of } e^{ix} \dfrac{1}{-2i}\left[1 - (1+i)D - \dfrac{1}{2i}D^2\right]^{-1} \cdot x$

$= \text{Imaginary part of } (\cos x + \sin x)\left(\dfrac{i}{2}\right)[1 + (1+i)D]x$

$= \text{Imaginary part of } \dfrac{1}{2}(i\cos x - \sin x)[x + 1 + i]$

P.I. $= \dfrac{1}{2}x\cos x + \dfrac{1}{2}\cos x - \dfrac{1}{2}\sin x$

Complete solution is $y = (C_1 + C_2 x)e^x + \dfrac{1}{2}(x\cos x + \cos x - \sin x)$

GENERAL METHOD OF FINDING THE PARTICULAR INTEGRAL OF ANY FUNCTION ϕ (x)

$$\frac{1}{D-a} \cdot \phi(x) = e^{ax}\int e^{-ax} \cdot \phi(x)dx$$

Cauchy-Euler Homogeneous Linear Differential Equations:

$$a_n x^n \frac{d^n y}{dx^n} + a_{n-1}x^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \ldots + a_0 y = \phi(x)$$

Where $a_0, a_1, a_2, \ldots$ are constants, is called a homogeneous equation.

Put $\quad x = e^z, \quad z = \log_e x, \quad \dfrac{d}{dz} = D$

$$\frac{dy}{dx} = \frac{dy}{dz}.\frac{dz}{dx} = \frac{1}{x}\frac{dy}{dz} \Rightarrow x\frac{dy}{dx} = \frac{dy}{dz} \Rightarrow x\frac{dy}{dx} = Dy$$

Again, $\quad \dfrac{d^2y}{dx^2} = \dfrac{d}{dx}\left(\dfrac{dy}{dx}\right) = \dfrac{d}{dx}\left(\dfrac{1}{x}\dfrac{dy}{dz}\right) = -\dfrac{1}{x^2}\dfrac{dy}{dz} + \dfrac{1}{x}\dfrac{d^2y}{dz^2}\dfrac{dz}{dx}$

$$= -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x}\frac{d^2y}{dz^2}\frac{1}{x} = \frac{1}{x^2}\left(\frac{d^2y}{dz^2} - \frac{dy}{dz}\right) = \frac{1}{x^2}(D^2 - D)y$$

$$x^2\frac{d^2y}{dx^2} = (D^2 - D)y$$

$$\Rightarrow \qquad \boxed{x^2 \frac{d^2y}{dx^2} = D(D-1)y}$$

Similarly. $\boxed{x^3 \frac{d^3y}{dx^3} = D(D-1)(D-2)\,y}$

Ques. Solve: $x^2 \frac{d^2y}{dx^2} - x\frac{dy}{dx} + 4y = \cos(\log x) + x\sin(\log x)$.

Sol. We have, $x^2 \frac{d^2y}{dx^2} - x\frac{dy}{dx} + 4y = \cos(\log x) + x\sin(\log x)$. …(1)

Putting $x = e^z \Rightarrow z = \log x, D = \frac{d}{dz}$ and $x^2 \frac{d^2y}{dx^2} = D(D-1)y, x\frac{dy}{dx} = Dy$ in (1), we get

$$[D(D-1) - D + 4]y = \cos z + e^z \sin z$$

i.e. $$(D^2 - 2D + 4)y = \cos z + e^z \sin z$$

A.E. is $m^2 - 2m + 4 = 0 \Rightarrow \qquad m = \frac{-(-2) \pm \sqrt{4-16}}{2}$

$$\Rightarrow \qquad m = 1 \pm \sqrt{3}\,i$$

$\therefore$ C.F. $= e^z[C_1 \cos\sqrt{3}\,z + C_2 \sin\sqrt{3}\,z]$ …(2)

P.I. $= \frac{1}{D^2 - 2D + 4}(\cos z + e^z \sin z)$

$$= \frac{1}{D^2 - 2D + 4}\cos z + \frac{1}{D^2 - 2D + 4} e^z \sin z$$

$$= \frac{1}{-1 - 2D + 4}\cos z + e^z \frac{1}{(D+1)^2 - 2(D+1) + 4}\sin z$$

$$= \frac{1}{3 - 2D}\cos z + e^z \frac{1}{D^2 + 2D + 1 - 2D - 2 + 4}\sin z$$

$$= \frac{3 + 2D}{9 - 4D^2}\cos z + e^z \frac{1}{D^2 + 3}\sin z = \frac{3 + 2D}{9 + 4}\cos z + e^z \frac{1}{-1 + 3}\sin z$$

$$= \frac{3 + 2D}{13}\cos z + e^z \frac{1}{2}\sin z = \frac{1}{13}(3\cos z - 2\sin z) + \frac{1}{2} e^z \sin z \quad \ldots(3)$$

Complete solution is

$$y = C.F. + P.I.$$

$$\Rightarrow \qquad y = e^z[C_1 \cos\sqrt{3}\, z + C_2 \sin\sqrt{3}\, z] + \frac{1}{13}(3\cos z - 2\sin z) + \frac{1}{2} e^z \sin z \qquad \ldots(4)$$

Replacing $z = \log x$ and $e^z = x$ in (4), we get

$$y = x[C_1 \cos\sqrt{3}(\log x) + C_2 \sin\sqrt{3}(\log x)]$$

$$+ \frac{3}{13}\cos(\log x) - \frac{2}{13}\sin(\log x) + \frac{1}{2} x \sin(\log x)$$